Formulation of Three-Dimensional Hodograph Method and Separable Solutions for Nonlinear Transonic Flows

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Formulation of a three-dimensional hodograph technique for transforming the full nonlinear potential transonic flow equation from the physical space into an equivalent linear counterpart in the hodograph plane is presented. A very careful examination of the governing nonlinear equations in the physical space reveals that a mild constraint on the energy equation (which may even enhance the accuracy of this nonviscous formulation) would permit the separation of the nonlinear flow equations for an aircraft wing into a sectional component and spanwise component. This separation of variables normally believed to be possible only for linear equations seems to have been possible (for the nonlinear equations) because of some inherent mathematical symmetry of the three-dimensional nonlinear flow equations. The consequential "three-dimensional" sectional equation is eventually transformed into the hodograph plane where it becomes linear. A further transformation of these linear hodograph equations into the characteristic hodograph plane provides the opportunity of obtaining the nonlinear flowfield for a particular set of boundary conditions by just solving a set of first order characteristic equations. The necessary computations can easily be carried out on a reasonably small computer. Some preliminary computations show good agreement with previously computed data. This verification therefore provides confidence that the new tool can perhaps be used in an inverse manner to design a new family of three-dimensional lifting surfaces with great potential.

Nomenclature

$\bar{a}, \bar{b}, \bar{d}, f_o$ = coefficients of hodograph equation
$B, \bar{B}$– = complex constant and its conjugate
C, C = speed of sound and two-dimensional component of three-dimensional speed of sound
C_i $(i = 1,2,\ldots)$ = coefficients of separated physical equations
$c_p, \bar{y}$ = pressure coefficient and fraction of wing half span (zero is wing root)
E, t = ellipse parameter and time
F = term in hodograph equation
f, h = functions in the hodograph plane
M = Mach number
$r, \hat{q}$ = ratio of specific heats and maximum speed
s, t = conjugate complex characteristics
x_i, z_i = regular solutions in the complex characteristic hodograph plane
u, v, w = flow velocity components
$\bar{u}, \bar{w}$ = two-dimensional components of three-dimensional flow velocities
$\bar{x}$ = fraction of chord length (zero is leading edge)
x, y, z = Cartesian coordinates (flow, spanwise, and vertical directions)
q, θ = flow speed and flow angle
Λ, α = spanwise component velocity potential and separation constant
ξ, η = characteristic coordinates
ρ, k = air density and circulation parameter
τ, λ = hodograph variable and characteristic roots
ϕ, ψ = velocity potential and stream function
Φ, χ = section component velocity potential and transformed potential
χ = transformed potential
$(\ )_\infty$ = freestream quantities

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Introduction

IN the late 1800s, certain investigators[1,2] discovered that the hodograph transformation can be a very powerful tool for studying nonlinear inviscid fluid flow problems. Subsequently, later investigators[3-14] were able to reveal that this method could be used to design supercritical airfoils with superior transonic performance characteristics. These developments constituted a breakthrough in the study of transonic flow problems, which are characterized by equations that exhibit strong nonlinearities. The success of this method was due to the following two characteristics of solutions obtained from the hodograph equations as compared to solutions obtained from the equations in physical space. First, the hodograph equations are linear and allow solutions to be superimposed, which is not possible in physical space due to the nonlinearity of the governing equations. Second, the potential for formulating an inverse problem in hodograph space thereby allowing the desired fluid flowfield to be specified as input and leading to the required geometry as the computed output.

Despite the success of this work, use of the hodograph technique has unnecessarily suffered from a major drawback. This drawback is a general belief that the hodograph technique can only be applied to two-dimensional steady flow problems. This implies that neither unsteady two-dimensional flow nor steady three-dimensional problems can be studied using a hodograph approach and thus has limited the interest in, and application of, this powerful analysis tool.

However, recent research performed by Oyibo[15] has shown that the hodograph transformation is not limited to two-dimensional steady flows. The work reported in this paper is a direct consequence of the discovery that, contrary to the aforementioned general belief, the hodograph technique can be employed to transform the nonlinear transonic flow problems with three independent variables into an equivalent linear set of equations in the hodograph plane.

The derivation of these transformed equations probably marks the first time in fluid mechanics that the hodograph technique has been extended beyond steady two-dimensional flow using full potential equations. It is important to note that a straightforward application of the usual hodograph (Legendre) transformation for the case of three independent variables leads to a hodograph equation,[15] which is even more nonlinear than its counterpart in physical space. While this conclusion is consistent with the earlier research work published by Guderley,[16] which was independently verified by Cole and Cook[17,21] using a different approach, small disturbance (not full potential) equations were used. The primary achievement of the new method was to find an alternate way of defining the transformation so as to result in a linear system of equations. The resulting equations have tremendous significance for the solution of both steady and unsteady fluid flow problems due to a number of features of the hodograph method in general and the new equations in particular.

First, since the transformed equations are linear, they allow superposition of solutions. This means that any complete solution can be constructed from a combination of fundamental solutions. This will enable an analytic determination of the influence of the input parameters on the solutions and hence will lead to a better understanding of the flow physics. This is not possible with the nonlinear equations in physical space where, except for specialized cases, only purely numerical solutions are possible. These numerical solutions suffer from the drawback that they are both approximate and that it is difficult to understand the effect that the important physical parameters have on the solution to the problem, since these effects have to be deduced by studying the results from a number of different cases.

Second, as shown below in Section 1.1, the form of the newly derived hodograph equations for both the unsteady two-dimensional case and the steady three-dimensional case are similar to the steady two-dimensional hodograph equations in that the highest order (second) derivative terms are identical for all three cases. The difference between the cases shows up in the inclusion of lower order terms that are not present in the steady two-dimensional hodograph equation. Since the computational solution techniques and the characteristics of partial differential equations are usually determined by the form of the highest derivative terms, and since that form is identical to the steady two-dimensional hodograph equations, then the solution techniques previously developed for steady two-dimensional flow should be directly applicable to the three-dimensional steady case. This is particularly important in the case of transonic flow due to the nonexistence of smooth solutions as proven by Morawetz.[20] For the case of two-dimensional steady flow, Garabedian and Korn[6] have worked out a systematic and efficient procedure to enable "closed" bodies to be developed in a straightforward manner. Since the new equations are so similar to the ones solved in Ref. 6, this work is directly applicable to the current problem and forms the bases for excluding nonrealistic or discontinuous solutions from the admissible set.

The final, and perhaps most important, point is that the transformed equations allow inverse solutions to be obtained, in which the pressure field is specified and the body geometry is calculated from closed form solutions. From the point of view of design, this is the desired situation rather than the usual trial and error method of picking the geometry and then examining the resulting calculated flowfield. Thus solutions obtained from the new hodograph method could lead to the design of new families of wing shapes that should prove to have significantly reduced shock strengths and hence lower drag when shocks appear at off design conditions. The fuel savings for the commercial airline industry made possible by such an efficient wing design could approach hundreds of millions of dollars annually and thus the proposed technique as outlined in this paper could lead to immediate and substantial industrial value.

In Ref. 15, the two-dimensional unsteady nonlinear transonic flow case was treated using the new hodograph method and shown to lead to results in good agreement with previous computational solutions. The ease of obtaining solutions to the new equations was demonstrated by using a microcomputer to obtain these solutions. This should allow more room in big computers like the Cray for design optimization rather than just solving for the flowfield.

Since that time, the derivation of the governing hodograph equations in three variables has been extended to the important case of steady three-dimensional flow. It is the purpose of this paper to delineate the formulation of the three-dimensional hodograph equations and to carry out a preliminary study of their solutions for the case of steady flow around lifting surfaces of finite span.

Three-Dimensional Hodograph Formulation

The three-dimensional formulation has been derived with the help of a persistent investigation and evolution of possible transformations that ultimately led to the revelation that the flow can be decomposed into a sectional component and another component in the spanwise direction. Perhaps it should be emphasized that the most important common thread between the two-dimensional unsteady and three-dimensional steady analyses is the search for a "mathematical symmetry" in the gasdynamic equations. Once this symmetry (which is not very obvious) is found, the use of a modified form of Legendre's transformation leads to a linear system of equations in hodograph space for both the two-dimensional unsteady and the three-dimensional steady cases. From the preliminary work that has been done, this mathematical symmetry concept seems to be also usable in investigating nonlinear hypersonic flow phenomena using Boltzmann's equations as well as modeling turbulence at hypersonic speeds.

For the case of three-dimensional nonlinear steady flow, the governing equations in physical space for the velocity potential ϕ is given by

$$(c^2-\phi_x^2)\phi_{xx}+(c^2-\phi_y^2)\phi_{yy}+(c^2-\phi_z^2)\phi_{zz}$$

$$-2\phi_x\phi_y\phi_{xy}-2\phi_x\phi_z\phi_{xz}-2\phi_y\phi_z\phi_{yz}=0 \tag{1}$$

Here again, as in the two-dimensional unsteady case, a straightforward application of Legendre's transformation in three variables to Eq. (1) leads to a highly nonlinear equation in hodograph space.

Following the analysis procedure developed in the two-dimensional unsteady work, the following alternate transformation

$$\chi = x\frac{\partial\phi}{\partial x}+z\frac{\partial\phi}{\partial z}-\phi \tag{2}$$

leads, after extensive algebraic manipulation, to the following linear equation in hodograph space:

$$(\bar{u}^2-\bar{c}^2)\bar{\chi}_{\bar{w}\bar{w}}+(\bar{w}^2-\bar{c}^2)\bar{\chi}_{\bar{u}\bar{u}}+2\bar{u}\,\bar{w}\bar{\chi}_{\bar{u}\bar{w}}+F=0 \tag{3}$$

with

$$\chi=\bar{\chi}(\alpha,x,z)\Lambda(\alpha,y) \tag{4}$$

where α is a parameter determined by the boundary conditions and Λ is the spanwise component of the flow solution. The χ and $\bar{\chi}$ are the transformed three-dimensional and two-dimensional potentials, respectively, and F is given by

$$F=-f_1\bar{\chi}_{\bar{u}}-f_2\bar{\chi}_{\bar{w}}+f_3\bar{\chi} \tag{5}$$

where

$$f_1=\frac{\bar{c}^2}{\bar{w}}\alpha^2\left\{1+2\left[\left(\frac{\bar{w}}{\bar{c}}\right)^2+\left(\frac{\bar{u}}{\bar{c}}\right)^2\right]\right\} \tag{6a}$$

$$f_2 = \frac{\bar{c}^2}{\bar{u}} \alpha^2 \left\{ 1 + 2\left[\left(\frac{\bar{w}}{\bar{c}}\right)^2 + \left(\frac{\bar{u}}{\bar{c}}\right)^2 \right] \right\} \tag{6b}$$

$$f_3 = \frac{\bar{c}^2}{\bar{u}\bar{w}} \alpha^2 \left\{ 1 + 2\left[\left(\frac{\bar{w}}{\bar{c}}\right)^2 + \left(\frac{\bar{u}}{\bar{c}}\right)^2 \right] \right\} \tag{6c}$$

$$w = \bar{w}(\alpha,x,z)\Lambda(\alpha,y), \; u = \bar{u}(\alpha,x,z)\Lambda(\alpha,y),$$

$$c = \bar{c}(\alpha,x,z)\Lambda(\alpha,y) \tag{7a}$$

To verify separability of solutions in the physical space, substitute Eq. (7a) and the following equation,

$$\phi = \Phi(\alpha,x,z)\Lambda(\alpha,y) \tag{7b}$$

into Eq. (1). The resulting equation is given by

$$f_4(-\Phi^3 f_5^2 + \bar{c}^2\Phi) - 2f_5^2(\bar{u}^2 + \bar{w}^2)\Phi - 2\bar{u}\bar{w}\Phi_{xz}$$
$$+ (\bar{c}^2 - \bar{u}^2)\Phi_{xx} + (\bar{c}^2 - \bar{w}^2)\Phi_{zz} = 0 \tag{7c}$$

where

$$f_4 = \Lambda_{yy}/\Lambda, \qquad f_5 = \Lambda_y/\Lambda$$

or

$$(\bar{c}^2 - \bar{u}^2)\Phi_{xx} + (\bar{c}^2 - \bar{w}^2)\Phi_{zz} - 2\bar{u}\bar{w}\Phi_{xz}$$
$$- \{2f_5^2(\bar{u}^2 + \bar{w}^2) - f_4\bar{c}^2\}\Phi - f_4 f_5^2 \Phi^3 = 0 \tag{7d}$$

Hence, for separability of solutions, f_4 and f_5 must be constants, or

$$\Lambda_{yy} - f_4\Lambda = 0, \qquad \Lambda_y - f_5\Lambda = 0 \tag{7e}$$

Equations (7d) and (7e) probably are among the very few (perhaps the first set) of separable nonlinear equations in fluid dynamics. It can be seen that as a consequence of Eq. (7a) either of the following constraints on the energy (or Bernoulli) equation become necessary:

$$\left(\frac{u_\infty^2}{2} + \frac{c_\infty^2}{\gamma - 1}\right) = 0 \text{ in three dimensions} \tag{7f}$$

or

$$(u_\infty, \; c_\infty) = (\bar{u}_\infty, \; \bar{c}_\infty)\Lambda(\alpha,y) \tag{7h}$$

For the constraints in Eq. (7f) to be enforced, γ must approach -1. This is either an interesting coincidence or something is being said about the realities of the isentropic assumptions in the potential flow formulations. This is because $\gamma = -1$ is the Karman-Tsien gas that has been shown to be so accurate that the results using this gas agree very well with experimental results. This agreement, which is really excellent, can be seen in Figure F,5b in Ref. 19. Figure F,5b further shows clearly that Karman-Tsien gas is very accurate even for truly transonic flows (flows with supersonic bubbles). The alternative constraint in Eq. (7h) could be shown to work provided that u_∞, c_∞ represent the reference section freestream velocity and speed of sound, respectively. Therefore, for steady three-dimensional flows, the hodograph equation to be solved is

$$(\bar{u}^2 - \bar{c}^2)\bar{\chi}_{\bar{w}\bar{w}} + (\bar{w}^2 - \bar{c}^2)\bar{\chi}_{\bar{u}\bar{u}} + 2\bar{u}\bar{w}\bar{\chi}_{\bar{u}\bar{w}} - f_1\bar{\chi}_{\bar{u}} - f_2\bar{\chi}_{\bar{w}} + f_3\bar{\chi} = 0 \tag{8}$$

If $\alpha = 0$, Eq. (8) reduces to the familiar two-dimensional hodograph equation.

Equation (8) is similar to the two-dimensional hodograph equation that has been studied for almost a century. Therefore any methods of solution for the two-dimensional hodograph equations are also applicable to this new equation. Due to the mathematical symmetry mentioned previously, the solution takes on the separated form given by Eq. (4) and Eq. (8) becomes the two-dimensional sectional analog of the three-dimensional flow problem. From Eq. (7e) the following forms are feasible solutions for the spanwise component

$$\Lambda(\alpha,y) = \prod_{n=0}^{\infty} \Lambda_n e^{\alpha_n y} \tag{9}$$

where α_n are constants determined by the boundary conditions of the flow and y is the spanwise coordinate.

The general solutions of the sectional component of Eq. (8) are in terms of hypergeometric series. These solutions may be used along with the method of complex characteristics in which the flow is mapped into the unit circle in the characteristic hodograph plane in order to obtain solutions for Eq. (8). The goal is then to obtain the body stream function that encloses the particular wing section (e.g., the root section). Thereafter the spanwise component is combined with this solution in accordance with Eq. (4) to provide the flowfield over the entire three-dimensional lifting surface.

Nonlinear Three-Dimensional Transonic Flow Investigation and Construction

The fundamental exact solutions to the new hodograph equations are hypergeometric series in the hodograph space. As mentioned above, the equation system in the hodograph plane closely resembles the two-dimensional steady hodograph case for which computational techniques are well established. It is expected that these solution techniques will therefore be directly transferable to the higher dimensional cases. The techniques that have been successfully applied in previous hodograph analyses fall generally into two categories, either solutions are obtained by superposition of a series of fundamental solutions as in the manner of Nieuwland[5] and as was also done in Ref. 15 or by converting the problem into an initial value problem (or a characteristic initial value problem) by using the method of complex characteristics developed by Garabedian.[10]

While either of the solution techniques can be used, the method of complex characteristics has more appeal since the location of sonic lines and limiting lines appear more naturally in characteristic coordinates and thus this method is expected to be more promising in this ongoing study. However, in the present study the series method way was used for constructing the flows satisfying the necessary boundary conditions for three-dimensional wings and to study the fundamental solutions carefully to determine how to piece them together properly as was done in Ref. 15 for the two-dimensional unsteady case. The details of this procedure are given in Ref. 15. An alternate method, which follows the two-dimensional work of Garabedian, is expected to be the primary analysis tool in the next stage of the study and consists of 1) reducing the linear hodograph equation into its canonical form, 2) utilizing the method of complex characteristics to map the particular section flow onto the unit circle in the characteristic hodograph plane, and 3) examining the solutions to ensure that the body stream function encloses the particular wing section.

For that purpose, Eq. (8) reduced to its canonical form would lead to the following system of first order equations

$$z_\xi + \lambda_+ x_\xi + \frac{F}{\bar{a}}\bar{w}_\xi = 0, \; z_\eta + \lambda_- x_\eta + \frac{F}{\bar{a}}\bar{w}_\eta = 0 \tag{10}$$

$$\bar{u}_\xi - \lambda_- \bar{w}_\xi = 0, \; \bar{w}_\eta - \lambda_+ \bar{u}_\eta = 0 \tag{11}$$

$$\chi_\xi - z\bar{w}_\xi - x\bar{u}_\xi = 0 \tag{12}$$

where ξ and η are the complex characteristics and

$$z = \chi_w,\ x = \chi_u$$

$$\lambda_\pm = \frac{\bar{b} \pm \sqrt{\bar{b}^2 - \bar{a}\bar{d}}}{\bar{a}} \tag{13}$$

with

$$\bar{a} = \bar{u}^2 - \bar{c}^2,\ \bar{b} = \bar{u}\bar{w},\ \bar{d} = \bar{w}^2 - \bar{c}^2 \tag{14}$$

Notice that the canonical system of Eqs. (10-14) are first order equations representing Eq. (8) or (3). One way to solve these equations is to map the flowfield in the physical space into the region inside a unit circle in the complex characteristic hodograph plane and then use the incompressible solution to provide initial data for solving the characteristic initial value problem resulting from the mapping. This approach (thoroughly documented in Ref. 5 is efficiently automated for the steady two-dimensional case in a computer code that we have obtained from Garabedian. We have extensively exercised this code and are thoroughly familiar with it, and hence have a major part of the necessary code for the three-dimensional problem in hand due to the similarity between the two-dimensional and three-dimensional equations. A full run of this code takes only 30 to 45 s on a Cray X-MP when run with nonvectorized code. It is expected that vectorization of the code that is currently under way will enable a fully optimized (shock-free) three-dimensional wing solution to be obtained in about 10 min, which appears to be significantly faster than the time current computational transonic three-dimensional codes would require to obtain an optimal flow.

In general either of the two methods outlined above can be examined and used. In that process, the fundamental solutions can be studied very carefully, particularly to ensure the proper handling of the singularities like the limiting lines, which are the zeros of the Jacobian of the transformation and the branch lines, where the Jacobian is infinite.

It is recognized that though the spanwise and sectional components of the flow are (in a sense) considered separately, the fundamental problem of shockless flow is basically to find smooth transonic solutions for Eq. (1), describing a three-dimensional compressible flow about a lifting surface of finite span. However, it is also recognized as shown by Morawetz,[20] that smooth transonic solutions do not even exist for all sectional two-dimensional shapes for the lifting surface (let alone for a three-dimensional lifting surface). The new hodograph transformation helps us to overcome this difficulty by making it possible to separate the three-dimensional flow into the two components mentioned above and solve them as an inverse problem instead. If the spanwise component is given by Eq. (9), then the sectional component is to be determined from Eqs. (10-14). In this study therefore, we computed smooth transonic flow by piecing together the hypergeometric series solutions to Eq. (8) as in Ref. 15 from a desired flowfield and find the finite span lifting surface that generates it. This procedure was used to find the necessary α and hence the spanwise variation of the flowfield. In order to compute the sectional component of this smooth transonic flow by the method of complex characteristics, the variables in Eqs. (10-14) can be extended into the complex domain where a characteristic initial value problem along the complex characteristics may be solved. While this procedure can be done in a purely analytic manner, it can also be carried out conveniently on a computer. In the future work, we propose to use the computational approach by modifying the existing code of Garabedian to include the additional terms appearing in the sectional equation, Eq. (8). When this is done, the "three-dimensional" section will be determined as well as α that provides the unknown in Eq. (9) for the spanwise component of the flow.

Considering Eq. (13), for real variables, for supersonic flow, λ_+ and λ_- are real and Eq. (8) is hyperbolic. This means that the initial value problem, defined as specifying x and z on any noncharacteristic curve as well as the characteristic initial value problem defined as one in which x or z is specified on one characteristic of each family are well posed. However, for subsonic flow, λ_+ and λ_- become complex conjugates and Eq. (8) is elliptical. Here, both the initial and the characteristic initial value problem are no longer well posed in the real domain, and hence boundary values are generally needed for formulating a correctly set problem.

Credibility of the New Hodograph Method

As has been mentioned earlier, the two-dimensional unsteady results have been shown, in Ref. 15 to be accurate when compared with previously published purely computational results.

For the three-dimensional steady case, Fig. 1 shows a comparison of preliminary results of the calculation of the flow from the new method with results obtained by Spreiter[18] using a local linearization technique for the case of $M = 1$ flow around a rectangular wing with a parabolic arc section and an aspect ratio of 7.61. The results are shown to be in good agreement, with the differences attributable to the approximate nature of Spreiter's analysis.

Figure 2 shows comparisons between the results obtained using the new method and those computed using the FLO22 computer code of Jameson and Caughey.[22] This code uses a finite difference technique to solve the transonic full potential equations in nonconservative form. For the cases shown of a rectangular wing at $M_\infty = 0.7$ the results are seen to be in excellent agreement due to the more accurate model contained in the computer code than in the previous case of Spreiter. We thus feel that these preliminary solutions give credibility to the three-dimensional analysis and give confidence as to its potential.

Relevance of Proposed Technique

Critics of the use, for design purposes, of the full potential equations (upon which the hodograph method is based) often

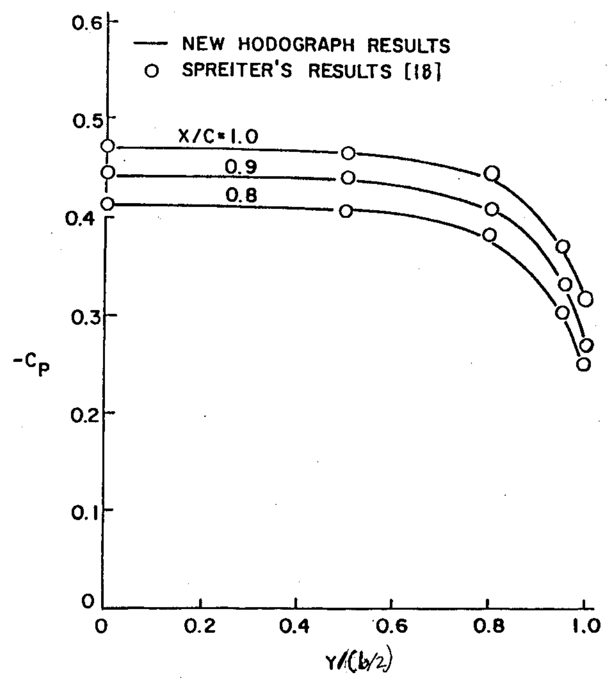

Fig. 1 Pressure coefficient vs spanwise station $M = 1.0$, $\alpha = 7.7$ degrees, rectangular wing aspect ratio = 7.63, 6% parabolic arc section.

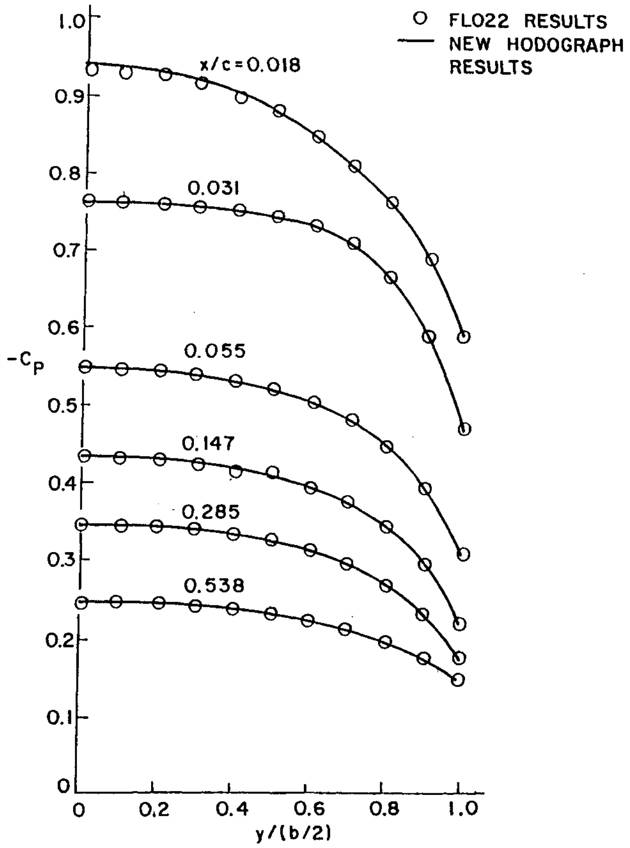

Fig. 2 Pressure coefficient vs spanwise station $M=0.7$, $\alpha=4$ degrees, rectangular wing aspect ratio = 3.2, NACA 65A006 section.

say that numerical solutions of either the three-dimensional Navier-Stokes equations or the Euler equations would provide a more accurate prediction of the flowfield, particularly at lower Reynolds numbers or in cases involving strong shocks. However, accurate numerical predictions by themselves do not necessarily provide design solutions, nor do they provide guidance to designers as to what to do about poor aerodynamic performance nor any other adverse conditions resulting from the predicted aerodynamic characteristics. In contrast, the solutions from the hodograph space do provide these capabilities. In addition, it is important to keep in mind that strong shocks are highly undesirable from a design point of view and should be avoided. Thus the capability of predicting strong shocks in the flowfield is not imperative for design point calculations. For the off-design condition, it is expected that for the aerodynamic shapes developed in this research, much weaker shocks are expected than for shapes designed based on two-dimensional analysis. For this situation solutions to the full potential equation are still valid and the simulation of shocks developed as part of the analysis work should be adequate to predict off design aerodynamic performance.

Note that it is generally believed that the discovery of supercritical wing sections was made possible by the use of the hodograph transformation along with the full potential flow equations (although classified experimental work was evolving the same airfoil shapes independently). As a result, new characteristic families of airfoil shapes were defined and, in conjunction with wind tunnel tests, were proven to be the types of shapes that can postpone or eliminate local shocks at transonic speeds.

In retrospect, it was not necessary to use a more accurate theory like the Navier-Stokes equations in order to discover such an important, fundamental and revolutionary result. In fact, it has been shown that the boundary layer correction effects can, in most cases, be satisfactorily incorporated empirically into the inviscid formulation (i.e., without necessarily using the Navier-Stokes equations). In addition, the difficulties of generating an appropriate grid and establishing the appropriate far-field boundary conditions and their location for the computational solution of the equations still are problems that require considerable numerical experimentation.

In spite of the successful design of shock free airfoils using the hodograph method, true shock free wings did not evolve from this work. One of the drawbacks of trying to use supercritical airfoils to construct three-dimensional lifting surfaces is the inherent significant sensitivity of the transonic flowfield to changes in its physical parameters due to the strong nonlinearity of the equations. For example, small changes in a given set of transonic flow parameters can result in a significantly different flowfield. This means, for instance, that a supercritical section designed to be shockless at a Mach number of 0.85 may end up having strong shocks at a Mach number of 0.8 or 0.9. This type of behavior is very typical of a nonlinear system (which is what the transonic flow problem is). The principle of superposition that is useful in linear systems does not apply to transonic flow. This, therefore, explains why a three-dimensional wing, designed with a set of two-dimensional sections along its span that are individually shockless, does not necessarily end up being a shockless wing. Unfortunately however, because the hodograph method (the principal tool for designing shockless sections) has been thought to be up until now a two-dimensional tool, designers have had no choice but to use the superposition idea to design a three-dimensional wing; this linearization practice (superposition) is fairly accurate in most flow regimes except the transonic. The penalty for such an inaccurate practice is the reappearance of strong shocks (and consequent deterioration of aerodynamic performance). Therefore, a three-dimensional hodograph approach should not only eliminate such an incorrect practice but should also provide the correct method for distributing the sectional properties (e.g., thickness) along the span providing perhaps new characteristic families of wing shapes that may provide truly shockless, three-dimensional transonic wings. It is emphasized that the availability of a design procedure based on the three-dimensional hodograph space transformation would provide insight not available from any other approach currently.

The treatment of shocks is very important for transonic flow. When the hodograph approach is used as an inverse method, the shock is not treated, since the goal in such an approach is to produce "shockless" airfoils. Some investigators are skeptical about the existence of "shockless" airfoils, mainly because of the possibility of the appearance of shocks at the off-design points. This is a genuine concern since the "shockless" airfoils that have been designed using a two-dimensional theory have to operate frequently at off-design points, e.g., operation in a three-dimensional or unsteady physical environment causing local off-design conditions even at the design Mach number. The ongoing work, which deals with application of the hodograph technique to the three-dimensional wing, should provide design points that are more practical and should alleviate such concerns.

Finally, this section may be concluded with a quotation from Sir M. J. Lighthill,[3] "It seems likely that any general theory of compressible flow applicable to problems with regions both of sub- and supersonic flow (such problems have been called 'trans-sonic') must be based on the 'hodograph transformation' (due originally to Molenbroek 1890 and Chaplygin 1904)."

Conclusions

In this paper the formulation of a new three-dimensional hodograph technique is presented. It is shown that a mild constraint on the energy/Bernoulli equation permits the separation of the nonlinear three-dimensional flow equations over an aircraft wing into the sectional component and spanwise component. These two component equations are solved in both physical space (spanwise component) and hodograph-plane (sectional component) and used for constructing flows over three-dimensional wings.

While the formulation of the complex characteristics method of solution for the sectional component is presented in this paper, the hypergeometric series solution method is utilized for constructing the flows whose results are shown in this preliminary computation work. A good agreement is seen when these results are compared to the previous work of Spreiter and FLO22 code developed by Jameson and Caughey. That verification of the present work provides the confidence that this new technique which seems to have significant potential also seems to possess some credibility.

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